J. DIFFERENTIAL GEOMETRY
4 (1970) 283-309

NEARLY KÄHLER MANIFOLDS

ALFRED GRAY

1. Introduction

Let M be a C^∞ almost Hermitian manifold with metric tensor $\langle\ ,\ \rangle$, Riemannian connection ∇, and almost complex structure J. Denote by $\mathscr{F}(M)$ the real valued C^∞ functions on M, and by $\mathfrak{X}(M)$ the C^∞ vector fields of M. Then M is said to be a *nearly Kähler manifold* provided $\nabla_X(J)(X) = 0$ for all $X \in \mathfrak{X}(M)$. Examples of nearly Kähler manifolds which are not Kählerian are S^6 (with the canonical almost complex structure and metric), and more generally G/K, where G is a compact semisimple Lie group and K is the fixed point set of an automorphism of G of order 3 (see [20]). If $\dim M \leq 4$, then M is Kählerian [12]. Thus we henceforth assume $\dim M \geq 6$.

A nearly Kähler manifold has the following property. Let $p \in M$ and let γ be a (piecewise differentiable) loop at p. Denote by τ_γ the parallel translation along γ, and let π be the holomorphic section of the tangent space of M at p which contains $\gamma'(0)$. Then there exists $g \in U(n)$ such that $\tau_\gamma | \pi = g | \pi$, where we regard $U(n)$ as the structure group of the tangent bundle of M. Conversely it is easy to see that any almost Hermitian manifold with this property is a nearly Kähler manifold. We say that $U(n)$ is a *weak holonomy group* of M. In a subsequent paper we shall investigate weak holonomy groups G for which G is transitive on some sphere. The most interesting situation occurs when $G = U(n)$.

We show in this paper that many well known theorems about the topology and geometry of Kähler manifolds can be generalized to nearly Kähler manifolds. The key fact is that the curvature operator $R_{XY}(X, Y \in \mathfrak{X}(M))$ of a nearly Kähler manifold satisfies certain identities described in § 2. These formulas resemble the corresponding formulas for Kähler manifolds sufficiently for us to carry over the proofs with a few changes.

In § 3 we generalize some formulas of [6] and [9] about holomorphic curvature to nearly Kähler manifolds. Furthermore, we define and discuss the properties of a particularly nice class of nearly Kähler manifolds, namely those of *constant type*. Pinching of nearly Kähler manifolds is discussed in § 4.

We observe in § 5 that a *compact nearly Kähler manifold of positive holomorphic sectional curvature is simply connected*. Furthermore *a complete*

Received March 17, 1969 and, in revised form, May 24, 1969.

nearly Kähler manifold whose holomorphic sectional curvature is positive and bounded away from zero is compact.

In § 6 and § 7 we discuss the cohomology groups $H^{p,q}(M)$ of a nearly Kähler manifold and generalize results of [2], [6], [7] and [9]. We prove in § 6 that for a non-Kählerian nearly Kähler manifold M whose sectional curvature satisfies a certain positivity condition we have $H^{1,1}(M) = 0$ (Theorem (6.2)). Then in § 7 we show that if the Ricci curvature of M satisfies a positivity condition we have $H^{p,0}(M) = H^{0,p}(M) = 0$ for $p > 0$ (Theorem (7.1)). Even though $H^{1,1}(M) = H^{2,0}(M) = H^{0,2}(M) = 0$ for a compact nearly Kähler manifold M, it is conceivable that $H^2(M, \boldsymbol{R}) \neq 0$. Nevertheless, we show that if a non-Kähler manifold has sufficiently large sectional curvature (or holomorphic pinching), we have $H^2(M, \boldsymbol{R}) = 0$.

§ 8 is devoted to the proof of the following generalization of a theorem of M. Berger [3]: *Let M be a compact Einstein nearly Kähler manifold of constant type. If M has positive sectional curvature and nonnegative holomorphic bisectional curvature, then either M is isometric to complex projective space or to S^6.* (See also [9].) In § 9 we determine differential forms which represent the Chern classes of a nearly Kähler manifold, or more generally any almost Hermitian manifold. Finally in § 10 we discuss immersions of nearly Kähler manifolds and generalize some results of [7], [8], and [11].

In connection with the results of § 8 we wish to make the following conjecture: *If M is a compact nearly Kähler manifold with positive sectional curvature, then M is isometric to complex projective space or to S^6.* It should at least be possible to prove this under the assumption that M has constant Ricci scalar curvature. For Kähler manifolds this was obtained in [5].

2. Curvature identities of nearly Kähler manifolds

In [11] we showed that the curvature operator $R_{XY}(X, Y \in \mathscr{X}(M))$ of a nearly Kähler manifold satisfies the identities

$$(1) \qquad \langle R_{XY}X, Y\rangle - \langle R_{XY}JX, JY\rangle = \|\nabla_X(J)(Y)\|^2 ,$$

$$(2) \qquad \langle R_{WX}Y, Z\rangle = \langle R_{JWJX}JY, JZ\rangle ,$$

for $W, X, Y, Z \in \mathscr{X}(M)$.

We give a generalization of formula (1) which will be useful.

Proposition (2.1). *For all $W, X, Y, Z \in \mathscr{X}(M)$ we have*

$$(3) \qquad \langle R_{WX}Y, Z\rangle - \langle R_{WX}JY, JZ\rangle = \langle \nabla_W(J)(X), \nabla_Y(J)(Z)\rangle .$$

Proof. Linearization of (1) together with the first Bianchi identity yields

$$(4) \qquad \begin{aligned} &3\langle R_{WX}Y, Z\rangle - \langle R_{WY}JX, JZ\rangle + \langle R_{WZ}JX, JY\rangle - 2\langle R_{WX}JY, JZ\rangle \\ &= \langle \nabla_W(J)(Y), \nabla_X(J)(Z)\rangle - \langle \nabla_W(J)(Z), \nabla_X(J)(Y)\rangle \\ &\quad + 2\langle \nabla_W(J)(X), \nabla_Y(J)(Z)\rangle . \end{aligned}$$

We replace Y and Z in (4) by JY and JZ and subtract the result from (4). Since $\nabla_U(J)(V) + \nabla_{JU}(J)(JV) = 0$ for all $U, V \in \mathscr{X}(M)$ [10], we obtain, after some simplification,

$$(5) \quad \begin{aligned} 4\langle \nabla_W(J)(X), \nabla_Y(J)(Z)\rangle &= 5\langle R_{WX}Y, Z\rangle \\ &- 5\langle R_{WX}JY, JZ\rangle - \langle R_{WJX}JY, Z\rangle - \langle R_{WJX}Y, JZ\rangle . \end{aligned}$$

In (5) we replace X and Y by JX and JY, and add $1/5$ of the resulting equation to (5). We then obtain (3).

Recently Goldberg and Kobayashi [9] have introduced the notion of *holomorphic bisectional curvature* for Kähler manifolds. Actually it is possible to define the holomorphic bisectional curvature B_{XY} for any almost Hermitian manifold. Thus if $X, Y \in \mathscr{X}(M)$, and $\|X\| \neq 0 \neq \|Y\|$, then we set

$$B_{XY}\|X\|^2\|Y\|^2 = \langle R_{XJX}Y, JY\rangle .$$

In particular, if M is nearly Kählerian, it follows from (1) that when X, JX, and Y are linearly independent,

$$(6) \quad B_{XY}\|X\|^2\|Y\|^2 = K_{XY}\|X \wedge Y\|^2 + K_{XJY}\|X \wedge JY\|^2 - 2\|\nabla_X(J)(Y)\|^2 ,$$

where K_{XY} denotes the sectional curvature of M of a field of 2-planes spanned by X and Y.

From (6) it follows that the *Ricci curvature* k of M is given by the formula

$$k(X, Y) = \sum_{i=1}^{n} \{\langle R_{XJY}E_i, JE_i\rangle + 2\langle \nabla_X(J)(E_i), \nabla_Y(J)(E_i)\rangle\} ,$$

where $X, Y \in \mathscr{X}(M)$ and $\{E_1, \cdots, E_n, JE_1, \cdots, JE_n\}$ is a frame field defined on an open subset of M.

The *holomorphic sectional curvature* $H(X)$ of M is defined by $H(X)\|X\|^4 = \langle R_{XJX}X, JX\rangle$ wherever $X \in \mathscr{X}(M)$ is nonzero. Obviously $H(X) = B_{XX}$. Also, for convenience we write $Q(X) = \langle R_{XJX}X, JX\rangle$ for $X \in \mathscr{X}(M)$. The *antiholomorphic sectional curvature* of M is the sectional curvature of M restricted to fields of 2-planes spanned by vector fields X and Y for which $\langle X, Y\rangle = \langle JX, Y\rangle = 0$.

Let $m \in M$, and denote by M_m the tangent space of M at m. Each of the tensor fields defined in this section gives rise to tensors on M_m which we denote by the same letters.

We next generalize some results of [4], [6].

Proposition 2.2. *Assume M is nearly Kählerian, and let $x, u \in M_m, m \in M$. Then*

$$\text{(i)} \quad \begin{aligned} \langle R_{xu}x, u\rangle = \frac{1}{32}\{&3Q(x + Ju) + 3Q(x - Ju) - Q(x + u) - Q(x - u) \\ &- 4Q(x) - 4Q(u)\} + \frac{3}{4}\|\nabla_x(J)(u)\|^2; \end{aligned}$$

(ii) $$\langle R_{xJx}u, Ju\rangle = \frac{1}{16}\{Q(x+Ju)+Q(x-Ju)+Q(x+u)+Q(x-u) - 4Q(x) - 4Q(u)\} - \frac{1}{2}\|\nabla_x(J)(u)\|^2 .$$

Proof. This is a verification using (1) and (2).

As an immediate consequence of these formulas, we obtain the following corollary.

Corollary (2.3). *Assume M is nearly Kählerian, and let $x, u \in M_m$ be such that $\|x\| = \|u\| = 1$ and $\langle x, u\rangle = \cos\varphi \geq 0$, $\langle x, Ju\rangle = \cos\theta \geq 0$. Then*

(i) $$K_{xu} = \frac{1}{8}\{3(1+\cos\theta)^2 H(x+Ju) + 3(1-\cos\theta)^2 H(x-Ju) - H(x+u) - H(x-u) - H(x) - H(u)\} + \frac{3}{4}\|\nabla_x(J)(u)\|^2, \text{ if } \langle x, u\rangle = 0 ;$$

(ii) $$B_{xu} = \frac{1}{4}\{(1+\cos\theta)^2 H(x+Ju) + (1-\cos\theta)^2 H(x-Ju) + (1+\cos\varphi)^2 H(x+u) + (1-\cos\varphi)^2 H(x-u) - H(x) - H(u)\} - \frac{1}{2}\|\nabla_x(J)(u)\|^2 .$$

Now let $x, u \in M_m$ be orthonormal vectors with $\langle x, Ju\rangle > 0$; then x and y span a plane Π in M_m. The average holomorphic curvature $H(x, u)$ and the average antiholomorphic curvature $A(x, u)$ of this plane (see [6]) are given by the formulas

$$H(x, u) = \frac{1}{\pi}\int_0^\pi H(x\cos\alpha + u\sin\alpha)d\alpha ,$$

$$A(x, u) = \frac{1}{\pi}\int_0^\pi K_{x\cos\alpha + u\sin\alpha, -Jx\sin\alpha + Ju\cos\alpha} d\alpha .$$

These formulas are independent of the choice of x and u in the plane Π.

Proposition 2.4. *If $x, u \in M_m$ are orthonormal, and $\langle x, Ju\rangle = \cos\theta \geq 0$, then we have*

$$\begin{aligned} K_{xu} &= H(x, u) - 3A(x, u)\sin^2\theta - 3\|\nabla_x(J)(u)\|^2 \\ &= \frac{1}{4}\{(1+\cos\theta)^2 H(x+Ju) + (1-\cos\theta)^2 H(x-Ju)\} \\ &\quad - A(x, u)\sin^2\theta - \frac{1}{2}\|\nabla_x(J)(u)\|^2 . \end{aligned}$$

The proof is a slight modification of a result of [4] for Kähler manifolds, and so we omit it.

3. Nearly Kähler manifolds of constant holomorphic curvature

We first prove the following result.

Proposition (3.1). *Let $x \in M_m$ be a unit vector at which the holomorphic sectional curvature H assumes its maximum at m. Then for all $y \in M_m$ with $\langle x, y\rangle = \langle Jx, y\rangle = 0$ and $\|y\| = 1$, we have*

$$H(x) \geq 3\langle R_{xy}x, y\rangle + \langle R_{xJy}x, Jy\rangle - 3\|\nabla_x(J)(y)\|^2 .$$

If H assumes its minimum at x, then the inequality is reversed.

Proof. Let a and b be real numbers with $a^2 + b^2 = 1$. A calculation shows that

$$\begin{aligned} H(ax + bJy) + H(ax - bJy) - 2a^4H(x) - 2b^4H(y) \\ = 4a^2b^2\{\langle R_{xJx}y, Jy\rangle + \langle R_{xy}x, y\rangle + \langle R_{xy}Jx, Jy\rangle\} \\ = 4a^2b^2\{3\langle R_{xy}x, y\rangle + \langle R_{xJy}x, Jy\rangle - 3\|\nabla_x(J)(y)\|^2\} . \end{aligned}$$

If H assumes its maximum at x, then

$$(1 - a^4)H(x) \geq b^4H(y) + 2a^2b^2\{3\langle R_{xy}x, y\rangle + \langle R_{xJy}x, Jy\rangle - 3\|\nabla_x(J)(y)\|^2\} ,$$

and so

$$(1 + a^2)H(x) \geq b^2H(y) + 2a^2\{3\langle R_{xy}x, y\rangle + \langle R_{xJy}x, Jy\rangle - 3\|\nabla_x(J)(y)\|^2\} .$$

We get the proposition by taking $a = 1$ in this equation.

Corollary (3.2). *Let x, y satisfy the hypotheses of Proposition* (3.1). *If the holomorphic sectional curvature H assumes its maximum at m, then*

$$H(x) \geq 2\langle R_{xJx}y, Jy\rangle + \|\nabla_x(J)(y)\|^2 .$$

If H assumes its minimum at x, then the inequality is reversed.

We now prove a related result which generalizes a result of [6].

Theorem (3.3). *Let M be an almost Hermitian manifold whose curvature operator satisfies* (2), *and assume that M has nonnegative holomorphic sectional curvature. Then the* 4*-dimensional sectional curvature $K_4(P)$ of a* 4*-dimensional subspace $P \subseteq M_m$ which is holomorphic* (i.e., *P is spanned by orthonormal vectors x, y, Jx, Jy*) *is nonnegative.*

Proof. Let $x \in P$ be a unit vector such that the holomorphic sectional curvature assumes its maximum on the unit sphere of P. Just as in [6] there exists $y \in P$ such that $\langle x, y\rangle = \langle x, Jy\rangle = 0$, $\|y\| = 1$, and $\langle R_{xy}x, Jy\rangle = \langle R_{Jxy}Jx, Jy\rangle = 0$, etc. Then the 4-dimensional sectional curvature $K_4(P)$ is a positive scalar multiple of

$$K_{xJx}K_{yJy} + K_{xy}^2 + K_{xJy}^2 + \langle R_{xJx}y, Jy\rangle^2 + \langle R_{xy}Jx, Jy\rangle^2 + \langle R_{xJy}Jx, y\rangle^2 .$$

Hence $K_4(P) \geq 0$.

Next we consider the case when the holomorphic sectional curvature $H(x)$ of a nearly Kähler manifold is constant.

Proposition (3.4). *Suppose the holomorphic sectional curvature H of M has the constant value μ at a point $m \in M$, and let $x, u \in M_m$ with $\|x\| = \|u\| = 1$. Then*

(i) $$K_{xu} = \frac{\mu}{4}\{1 + 3\langle Jx, u\rangle^2\} + \frac{3}{4}\|\nabla_x(J)(u)\|^2 , \qquad \text{if } \langle x, u\rangle = 0 ;$$

(ii) $$B_{xu} = \frac{\mu}{2}\{1 + \langle x, u\rangle^2 + \langle Jx, u\rangle^2\} - \frac{1}{2}\|\nabla_x(J)(u)\|^2 .$$

Proof. Write $u = ax + bJx + cy$ where $\|y\| = 1$ and $\langle x, y\rangle = \langle Jx, y\rangle = 0$; then $a^2 + b^2 + c^2 = 1$. Since H is contant at m, $\langle R_{xJx}x, y\rangle = \langle R_{xJx}Jx, y\rangle = 0$. Also, by Proposition (3.1) we have

$$\langle R_{xy}x, y\rangle = \langle R_{xJy}x, Jy\rangle = \frac{1}{4}\{\mu + 3\|\nabla_x(J)(y)\|^2\} .$$

Therefore

$$\begin{aligned}\langle R_{xu}x, u\rangle &= b^2H(x) + c^2\langle R_{xy}x, y\rangle \\ &= b^2\mu + \frac{c^2}{4}\{\mu + 3\|\nabla_x(J)(y)\|^2\} \\ &= \frac{\mu}{4}(1 - a^2 + 3b^2) + \frac{3}{4}\|\nabla_x(J)(u)\|^2 \qquad (7) \\ &= \frac{\mu}{4}(1 - \langle x, u\rangle^2 + 3\langle Jx, u\rangle^2) + \frac{3}{4}\|\nabla_x(J)(u)\|^2 .\end{aligned}$$

Hence (i) and (ii) follow easily from (7).

The following notions will be useful.

Definitions. Let M be an almost Hermitian manifold. Then M is said to be of *constant type* at $m \in M$ provided that for all $x \in M_m$ we have $\|\nabla_x(J)(y)\| = \|\nabla_x(J)(z)\|$ whenever $\langle x, y\rangle = \langle Jx, y\rangle = \langle x, z\rangle = \langle Jx, z\rangle = 0$ and $\|y\| = \|z\|$. If this holds for all $m \in M$ we say that M has (*pointwise*) *constant type*. Finally, if M has pointwise constant type and for $X, Y \in \mathscr{X}(M)$ with $\langle X, Y\rangle = \langle JX, Y\rangle = 0$ the function $\|\nabla_X(J)(Y)\|$ is constant whenever $\|X\| = \|Y\| = 1$, then we say that M has *global constant type*.

The proof of the following proposition is easy, and so we omit it.

Proposition (3.5). *Let M be a nearly Kähler manifold. Then M has (pointwise) constant type if and only if there exists $\alpha \in \mathscr{F}(M)$ such that*

$$(8) \qquad \|\nabla_W(J)(X)\|^2 = \alpha\{\|W\|^2\|X\|^2 - \langle W, X\rangle^2 - \langle W, JX\rangle^2\}$$

for all $W, X \in \mathscr{X}(M)$. Furthermore, M has global constant type if and only if (8) *holds with a constant function α.*

We agree to call α in (8) the *constant type* of M. It is unknown to the author whether if M has pointwise constant type then M has global constant type. A similar statement applies to holomorphic sectional curvature. The only example known to the author of a nearly Kähler manifold of constant holomorphic sectional curvature which is not Kählerian is S^6.

Proposition (3.6). *Let M be a nearly Kähler manifold with pointwise constant holomorphic sectional curvature μ and pointwise constant type α. Then*

(i) *M is an Einstein manifold with* $4k(x, x) = (n + 3)\mu + 3(n - 1)\alpha$, *where x is a unit vector and* $\dim = 2n$,

(ii) *at each point M has constant antiholomorphic sectional curvature* $(\mu + 3\alpha)/4$.

Proof. This follows easily from proposition (3.4).

4. Pinching of nearly Kähler manifolds

In this section we generalize the results of several authors [2], [4], [5], [6] to nearly Kähler manifolds. The curvature of a nearly Kähler manifold can be pinched in at least six different ways. In order to describe these, we first consider a sequence of conditions. Let $0 \leq \eta \leq 1$, and let M be a nearly Kähler manifold. For each of the conditions listed below, L denotes some number depending on M and η.

$R(\eta)$: $\eta L < K_{xu} \leq L$ for linearly independent $x, u \in M_m$ for all $m \in M$.

$H(\eta)$: $\eta L < K_{xJx} \leq L$ for nonzero $x \in M_m$ for all $m \in M$.

$BH(\eta)$: $\eta L < B_{xu} \leq L$ for nonzero $x, u \in M_m$ for all $m \in M$.

$BS(\eta)$: $\eta L < K_{xu} + K_{Jxu} \leq L$ for linearly independent $x, u, Jx, Ju \in M_m$ for all $m \in M$.

$K(\eta)$: $$\eta L(1 + 3\langle Jx, u\rangle^2) < K_{xu} - \frac{3}{4}\|\nabla_x(J)(u)\|^2 \leq L(1 + 3\langle Jx, u\rangle^2)$$

for $x, u \in M_m$ with $\|x\| = \|u\| = 1$, $\langle x, u\rangle = 0$, for all $m \in M$.

$BK(\eta)$: $$\eta L(1 + \langle x, u\rangle^2 + \langle Jx, u\rangle^2) < B_{xu} + \frac{1}{2}\|\nabla_x(J)(u)\|^2 \leq L(1 + \langle x, u\rangle^2 + \langle Jx, u\rangle^2)$$

for $x, u \in M_m$ with $\|x\| = \|u\| = 1$ for all $m \in M$.

For $C = R, H, BH, BS, K, BK$ we say that M is *δ C-pinched* if and only if $\delta = \text{lub}\{\eta \mid C(\eta) \text{ holds}\}$. Here R stands for Riemannian, H for holomorphic,

BH for biholomorphic, BS for bisectional, K for Kählerian, and BK for bi-Kählerian.

One problem with these pinchings is to determine the relations among them; another is to obtain bounds for the Ricci and Ricci scalar curvature in terms of the various pinchings. We determine some of these relations and bounds. For later applications we shall be particularly interested in determining the values of δ for which a δ holomorphically pinched manifold has nonnegative biholomorphic pinching and positive bisectional pinching.

It is clear that since we are dealing with nearly Kähler manifolds, the size of $\|\nabla_x(J)(y)\|^2$ will be important in the pinching estimates. For this reason we shall say that a nearly Kähler manifold *satisfies condition* $T(\rho,\sigma)$ provided that $\rho H(x) \leq \|\nabla_x(J)(y)\|^2 \leq \sigma H(x)$ for $x, y \in M_m$ with $\|x\| = \|y\| = 1$, $\langle x, y\rangle = \langle Jx, y\rangle = 0$ for all $m \in M$.

Proposition (4.1). *Let M be a nearly Kähler manifold which satisfies condition $T(\rho,\sigma)$. If M has nonnegative holomorphic bisectional curvature, then* $0 \leq \rho \leq \sigma \leq 1$.

Proof. This is a consequence of Corollary (3.2).

We first determine bounds on the sectional curvature in terms of holomorphic pinching. Assume in Propositions (4.2)–(4.4) that M has holomorphic pinching δ and that $\delta L \leq H(x) \leq L$ for nonzero $x \in M_m$ for all $m \in M$. Also suppose that condition $T(\rho,\sigma)$ is satisfied. Denote by $x, u \in M_m$ orthonormal vectors with $\langle x, Ju\rangle = \cos\theta > 0$, and let x, Jx, y, Jy be orthonormal.

Proposition (4.2). (i) *We have*

$$\frac{3}{8}(2\delta + 2\delta\cos^2\theta - 1)L + \left(-\frac{1}{8} + \frac{3}{4}\rho\sin^2\theta\right)H(x)$$
$$\leq K_{xu} \leq \frac{3}{8}(2 + 2\cos^2\theta - \delta)L + \left(-\frac{1}{8} + \frac{3}{4}\sigma\sin^2\theta\right)H(x)\ .$$

(ii) *If $\sigma\sin^2\theta \leq 1/6$, then*

$$\left(\frac{3}{4}\delta + \frac{3}{4}\delta\cos^2\theta - \frac{1}{2} + \frac{3}{4}\rho\sin^2\theta\right)L$$
$$\leq K_{xu} \leq \left(\frac{3}{4} + \frac{3}{4}\cos^2\theta - \frac{1}{2}\delta + \frac{3}{4}\delta\sigma\sin^2\theta\right)L\ .$$

(iii) *If $\rho\sin^2\theta \leq 1/6 \leq \sigma\sin^2\theta$, then*

$$\left(\frac{3}{4}\delta + \frac{3}{4}\delta\cos^2\theta - \frac{1}{2} + \frac{3}{4}\rho\sin^2\theta\right)L$$
$$\leq K_{xu} \leq \left(\frac{5}{8} + \frac{3}{4}\cos^2\theta - \frac{3}{8}\delta + \frac{3}{4}\sigma\sin^2\theta\right)L\ .$$

(iv) *If* $\rho \sin^2\theta \geq 1/6$, *then*

$$\left(\frac{5}{8}\delta + \frac{3}{4}\delta\cos^2\theta - \frac{3}{8} + \frac{3}{4}\delta\rho\sin^2\theta\right)L \leq K_{xu} \leq \left(\frac{5}{8} + \frac{3}{4}\cos^2\theta - \frac{3}{8}\delta + \frac{3}{4}\sigma\sin^2\theta\right)L\,.$$

Proof. (i) follows from Corollary (2.3), and (ii), (iii), and (iv) follow from (i).

Corollary (4.3). (i) *We have*

$$\left(\frac{3}{4}\delta - \frac{3}{8}\right)L + \left(-\frac{1}{8} + \frac{3}{4}\rho\right)H(x) \leq K_{xy} \leq \left(\frac{3}{4} - \frac{3}{8}\delta\right)L + \left(-\frac{1}{8} + \frac{3}{4}\sigma\right)H(x)\,.$$

(ii) *If* $\sigma \leq 1/6$, *then*

$$\left(\frac{3}{4}\delta - \frac{1}{2} + \frac{3}{4}\rho\right)L \leq K_{xy} \leq \left(\frac{3}{4} - \frac{1}{2}\delta + \frac{3}{4}\sigma\delta\right)L\,.$$

(iii) *If* $\rho \leq 1/6 \leq \sigma$, *then*

$$\left(\frac{3}{4}\delta - \frac{1}{2} + \frac{3}{4}\rho\right)L \leq K_{xy} \leq \left(\frac{5}{8} - \frac{3}{8}\delta + \frac{3}{4}\sigma\right)L\,.$$

(iv) *If* $\rho \geq 1/6$, *then*

$$\left(\frac{5}{8}\delta - \frac{3}{8} + \frac{3}{4}\rho\delta\right)L \leq K_{xy} \leq \left(\frac{5}{8} - \frac{3}{8}\delta + \frac{3}{4}\sigma\right)L\,.$$

We give another set of bounds for the sectional curvature which sometimes are better than those of Proposition (4.2).

Proposition (4.4). (i) *We have*

$$\left(\delta - \left(\frac{3}{4} + \frac{1}{8}\delta\right)\sin^2\theta\right)L + \left(\frac{1}{8} - \frac{5}{4}\sigma\right)\sin^2\theta\, H(x) \leq K_{xu} \leq \left(1 - \left(\frac{1}{8} + \frac{3}{4}\delta\right)\sin^2\theta\right)L + \left(\frac{1}{8} - \frac{5}{4}\rho\right)\sin^2\theta\, H(x)\,.$$

(ii) *If* $\sigma \leq 1/10$, *then*

$$\left(\delta - \frac{1}{4}(3 + 5\sigma\delta)\sin^2\theta\right)L \leq K_{xu} \leq \left(1 - \frac{1}{4}(3\delta + 5\rho)\sin^2\theta\right)L\,.$$

(iii) *If* $\rho \leq 1/10 \leq \sigma$, *then*

$$\left(\delta - \frac{1}{8}(5 + \delta + 10\sigma)\sin^2\theta\right)L \leq K_{xu} \leq \left(1 - \frac{1}{4}(3\delta + 5\rho)\sin^2\theta\right)L .$$

(iv) *If* $\rho \geq 1/10$, *then*

$$\left(\delta - \frac{1}{8}(5 + \delta + 10\sigma)\sin^2\theta\right)L$$
$$\leq K_{xu} \leq \left(1 - \frac{1}{8}(1 + 5\delta + 10\rho\delta)\sin^2\theta\right)L .$$

Proof. (i) follows from Proposition (2.4) and Corollary (4.3); (ii), (iii), and (iv) follow from (i).

Corollary (4.5). *If the holomorphic sectional curvature of a nearly Kähler manifold M is nonnegative, then at each point of M, a maximum sectional curvature is holomorphic.*

This generalizes [6, Theorem 8.2].

Next we obtain bounds on the holomorphic bisectional curvature in terms of holomorphic pinching. We use the conventions of Propositions (4.2)–(4.4), except that we assume that $\|x\| = \|u\| = 1$ and $\langle x, u\rangle = \cos\varphi > 0$, $\langle x, Ju\rangle = \cos\theta > 0$. These new conventions remain in effect throughout the rest of §4.

Proposition (4.6). (i) *We have*

$$\left(\delta - \frac{1}{4} + \frac{1}{2}\delta(\cos^2\theta + \cos^2\varphi)\right)L - \left(\frac{1}{4} + \frac{\sigma}{2} - \frac{1}{2}(\cos^2\varphi + \cos^2\theta)\sigma\right)H(x)$$
$$\leq B_{xu} \leq \left(1 - \frac{1}{4}\delta + \frac{1}{2}(\cos^2\theta + \cos^2\varphi)\right)L$$
$$- \left(\frac{1}{4} + \frac{\rho}{2} - \frac{1}{2}(\cos^2\varphi + \cos^2\theta)\rho\right)H(x) .$$

(ii)
$$\left(\delta - \frac{1}{2} - \frac{\sigma}{2} + \frac{1}{2}(\delta + \sigma)(\cos^2\varphi + \cos^2\theta)\right)L$$
$$\leq B_{xu} \leq \left(1 - \frac{1}{2}\delta - \frac{1}{2}\rho\delta + \frac{1}{2}(1 + \rho\delta)(\cos^2\varphi + \cos^2\theta)\right)L .$$

Proof. (i) follows from Corollary (2.3), and (ii) from (i).

Corollary (4.7). *If M is a nearly Kähler manifold for which the holomorphic pinching* $\delta \geq \frac{1}{2}(1 + \sigma)$, *then M has nonnegative holomorphic bisectional curvature.*

Proof. From Proposition (4.6) we have

$$B_{xu} \geq \delta - 1/2 - \sigma/2 > 0 .$$

Proposition (4.8). *Suppose M is a nearly Kähler manifold with holomorphic pinching δ which satisfies*

$$\delta > 1/(3\rho + 2)\ , \qquad \textit{if } \rho \leq 1/3\ ,$$
$$\delta > 1/(6\rho + 2)\ , \qquad \textit{if } \rho \geq 1/3\ .$$

Then $K_{xu} + K_{Jxu} > 0$, i.e., M has positive bisectional pinching.

Proof. We have

$$\begin{aligned} K_{xu} + K_{Jxu} &\geq \sin^2 \varphi K_{xu} + \sin^2 \theta K_{Jxu} \\ &= B_{xu} + 2\|\nabla_x(J)(u)\|^2 \\ &\geq \left(\delta - \frac{1}{2} + \frac{3}{2}\rho\delta\right)L + \frac{1}{2}\delta L(1 - 3\rho)(\cos^2\theta + \cos^2\varphi)\ . \end{aligned} \tag{9}$$

If $\rho \leq 1/3$, the right hand side of (9) is nonnegative, and so in this case, $K_{xu} + K_{xJu} \geq \left(\delta - \frac{1}{2} + \frac{3}{2}\rho\delta\right)L$. Hence $\delta > 1/(3\rho + 2)$ implies $K_{xu} + K_{Jxu} > 0$. On the other hand, if $\rho \geq 1/3$, then $K_{xu} + K_{Jxu} \geq \left(\delta - \frac{1}{2} + \frac{3}{2}\rho\delta\right)L + \frac{1}{2}\delta L(3\rho - 1) = \left(\frac{1}{2}\delta - \frac{1}{2} + 3\rho\delta\right)L$. Thus if $\delta > 1/(6\rho + 2)$ we again obtain $K_{xu} + K_{xJu} > 0$.

Corollary (4.9). *Let M be a nearly Kähler manifold which satisfies condition $T(\rho, \sigma)$ with $\sigma > 0$. If M has holomorphic pinching $\delta \geq \frac{1}{2}(1 + \sigma)$, then $B_{xu} \geq 0$ for nonzero $x, u \in M_m$, and $K_{xu} + K_{Jxu} > 0$ for linearly independent $x, u, Jx, Ju \in M_m$.*

Proof. We may take $\rho = 0$ in Proposition (4.8). The corollary now follows from Propositions (4.7) and (4.8).

We next obtain bounds on the Ricci curvature of a nearly Kähler manifold in terms of the holomorphic pinching. These will be useful in § 7. First we need the following estimates.

Proposition (4.10). *We have*

$$\begin{aligned} &\left(\delta - \frac{1}{4} + \frac{3}{2}\rho\delta\right)L - \frac{1}{4}H(x) \\ &\qquad \leq K_{xy} + K_{xJy} \leq \left(1 - \frac{1}{4}\delta + \frac{3}{2}\sigma\right)L - \frac{1}{4}H(x)\ . \end{aligned}$$

Proof. This is a consequence of Corollary (2.3).

We now estimate the Ricci curvature.

Proposition (4.11). *Let $x \in M_m$ be a unit vector and* $\dim M = 2n$. *Then*

$$\frac{L}{4}\{-(n-1)+4(n-1)\delta+6(n-1)\rho\delta\}-\frac{1}{4}(n-5)H(x)$$
$$\leq k(x,x)\leq\frac{L}{4}\{-(n-1)\delta+4(n-1)+6(n-1)\sigma\}-\frac{1}{4}(n-5)H(x)\ .$$

Proof. This is an easy consequence of Proposition (4.10).

Proposition (4.12). *Let* $\dim M = n$.

(i) *If* $n \leq 5$, *then*

$$\frac{1}{4}\{(3n+1)\delta+6(n-1)\rho\delta-(n-1)\}L$$
$$\leq k(x,x)\leq\frac{1}{4}\{(3n+1)-(n-1)\delta+6(n-1)\sigma\}L\ .$$

(ii) *If* $n > 5$, *then*

$$\left\{(n-1)\delta+\frac{3}{2}(n-1)\rho\delta-\frac{1}{2}(n-3)\right\}L$$
$$\leq k(x,x)\leq\left\{(n-1)+\frac{3}{2}(n-1)\sigma-\frac{1}{2}(n-3)\delta\right\}L\ .$$

Proof. This follows from Proposition (4.11).

We now prove some results about Riemannian pinching of nearly Kähler manifolds.

Proposition (4.13). *Suppose M is a nearly Kähler manifold which satisfies condition* $T(\rho,\sigma)$, *and assume M has Riemannian pinching* λ. *Then* $\lambda \leq \frac{1}{4}(1+3\sigma)$.

Proof. We may normalize the metric of M so that $\lambda \leq K_{xu} \leq 1$ for all linearly independent $x, u \in M_m$ for all $m \in M$. Let $x, y, Jx, Jy \in M_m$ be orthonormal. A result of [1] implies that $|\langle R_{xJx}y, Jy\rangle| \leq 2(1-\lambda)/3$. Hence $\lambda \leq K_{xy} + K_{xJy} \leq 2/3 - 5\lambda/3 + 2\sigma$ and so $\lambda \leq (1+3\sigma)/4$.

Proposition (4.14). *Suppose M is a nearly Kähler manifold which satisfies condition* $T(\rho,\sigma)$, *and assume M has Riemannian pinching* $\lambda < 1$ *and holomorphic pinching* δ. *Then*

$$\delta \geq (\lambda + 8\lambda^2 + 9\sigma^2 - 18\lambda\sigma)/(1-\lambda)\ .$$

Proof. We normalize the metric of M as in Proposition (4.13). Let $x, y, Jx, Jy \in M$ be orthonormal. A result of [2] which is valid for all Riemannian manifolds implies that

$$3\langle R_{xJx}y, Jy\rangle \leq 2(K_{xJx}-\lambda)^{1/2}(K_{yJy}-\lambda)^{1/2}+K_{xy}+K_{xJy}-2\lambda\ ,$$

and so

$$K_{xy} + K_{xJy} + \lambda - 3\|\nabla_x(J)(y)\|^2 \leq (K_{xJx} - \lambda)^{1/2}(K_{yJy} - \lambda)^{1/2} .$$

Since $K_{xy} \geq \lambda$, $K_{xJy} \geq \lambda$, $K_{yJy} \leq 1$, and $\|\nabla_x(J)(y)\|^2 \leq \sigma$, we have

$$3(\lambda - \sigma) \leq (K_{xJx} - \lambda)^{1/2}(1 - \lambda)^{1/2} ,$$

from which follows the proposition.

Proposition (4.15). *Suppose M is a nearly Kähler manifold with constant positive holomorphic sectional curvature μ aud nonnegative holomorphic bisectional curvature. Also assume that M satisfies condition $T(\rho, \sigma)$. If λ denotes the Riemannian pinching of M, then*

$$4\lambda \geq 1 + 3\rho .$$

Hence if $\rho > 0$ and M is compact, then M is homeomorphic to S^6.

Proof. Let $m \in M$, and let $x, u \in M_m$ be orthonormal. Write $\cos^2\theta = \langle Jx, u\rangle^2$. Then by Proposition (3.4) we have

$$\frac{\mu}{4}\{1 + 3(\cos^2\theta + \rho\sin^2\theta)\} \leq K_{xu} \leq \frac{\mu}{4}\{1 + 3(\cos^2\theta + \sigma\sin^2\theta)\} .$$

Since $0 \leq \rho \leq \sigma \leq 1$, we have

$$\mu(1 + 3\rho)/4 \leq K_{xu} \leq \mu .$$

Hence the proposition follows.

5. Simple connectivity and compactness of nearly Kähler manifolds

In this section we generalize some results of Tsukamoto [18]. The proofs are essentially the same as those for Kähler manifolds.

Theorem (5.1). *Let M be a compact nearly Kähler manifold of positive holomorphic sectional curvature. Then M is simply connected.*

Proof. Assume the contrary. Then there exists a non-trivial free homotopy class of loops which contains a non-trivial minimal geodesic σ. We may assume that σ has unit speed and is defined on $[0, b]$. Denote by σ' the velocity vector of σ. Since M is nearly Kählerian, $J\sigma'$ is parallel on σ. The deformation of σ given by $J\sigma'$ has second variation

$$I(J\sigma', J\sigma') = -\int_0^b K_{\sigma' J\sigma'}(t)dt < 0 .$$

Thus σ' cannot be a minimal geodesic. Hence M is simply connected.

It would be interesting to know if in Theorem (5.1) the assumption of positive holomorphic curvature could be replaced by that of positive Ricci curvature. This would be a natural generalization of a result of Kobayashi [15].

We remark that there exist compact non-simply connected nearly Kähler manifolds with nonnegative sectional curvature which are not Kählerian. An example is $P^7 \times P^7$ (see [20]), where P^7 denotes the real 7-dimensional projective space. On the other hand, in [19] it is shown that compact homogeneous almost complex manifolds of positive Euler characteristic are simply connected.

Theorem (5.2). *Let M be a complete nearly Kähler manifold whose holomorphic sectional curvature satisfies $K_{xJx} \geq \delta > 0$ for $x \in M_m$ and all $m \in M$. Then M is compact and the diameter of M is not greater than $\pi/\sqrt{\delta}$.*

Proof. Let $p, q \in M$. Since M is complete, there exists a unique unit speed geodesic σ defined on $[0, b]$ from p to q. Then $J\sigma'$ is parallel on σ. Let X be the vector field on σ defined by $X(t) = \left(\sin \frac{\pi t}{b}\right) J\sigma'(t)$. The deformation of σ given by X has second variation

$$\begin{aligned} I(X, X) &= \int_0^b \{\|X'\|^2 - \langle R_{X\sigma'}X, \sigma'\rangle\}(t)dt \\ &\leq \int_0^b \left\{\frac{\pi^2}{b^2}\cos^2\frac{\pi t}{b} - \delta \sin^2\frac{\pi t}{b}\right\} dt \\ &\leq b(\pi^2/b^2 - \delta) . \end{aligned}$$

Hence, if $b > \frac{\pi}{\sqrt{\delta}}$, then $I(X, X) < 0$, and so σ has a conjugate point. Since any two points are connected by a unique geodesic, the theorem follows.

6. Harmonic forms on nearly Kähler manifolds

Let M be any manifold with an almost complex structure J. We can decompose $\mathscr{X}(M) \otimes \boldsymbol{C}$ as

$$\begin{aligned} \mathscr{X}(M) \otimes \boldsymbol{C} &= \mathscr{I}_{+1} \oplus \mathscr{I}_{-1} \quad \text{where} \\ \mathscr{I}_{\pm 1} &= \{X \in \mathscr{X}(M) \otimes \boldsymbol{C} \mid JX = \pm iX\} . \end{aligned}$$

Definition. Let ω be a differential form (possibly complex) on an almost complex manifold M. Then ω is said to be of *bidegree* (p, q) if and only if ω is of degree $p + q$ and $\omega(X_1, \cdots, X_{p+q}) = 0$ whenever more than p of the X_j are in $\mathscr{I}_{-1}$ or more than q of the X_j are in $\mathscr{I}_{+1}$.

This generalization of the notion of (p, q) forms from complex manifolds to almost complex manifolds is due to Kozul [17]. If ω is a (p, q) form, then in general $d\omega$ has components which are $(p - 1, q + 2)$, $(p, q + 1)$, $(p + 1, q)$, and $(p + 2, q - 1)$ forms.

We shall need the following lemma.

Lemma (6.1). *Let ξ be a form of degree p on an almost complex manifold M.*

(i) *Suppose ξ has degree 2. Then ξ has bidegree (1, 1) if and only if $\xi(JX, JY) = \xi(X, Y)$ for all $X, Y \in \mathscr{X}(M)$.*

(ii) *ξ is the sum of forms of bidegrees $(p, 0)$ and $(0, p)$ if and only if $\xi(JX, JY, X_3, \cdots, X_p) = -\xi(X, Y, X_3, \cdots, X_p)$ for all $X, Y, X_3, \cdots, X_p \in \mathscr{X}(M)$.*

The proof is easy and we omit it.

Let $\Lambda^{p,q}(M)$ denote the complex differential forms of bidegree (p, q) on M, and $H^p(M)$ the space of real harmonic forms of degree p. We set

$$H^{p,q}(M) = (H^{p+q}(M) \otimes C) \cap \Lambda^{p,q}(M) .$$

It is known that if M is a Kähler manifold, then $H^{1,1}(M)$ is 1-dimensional if M has positive sectional curvature [7] (or positive holomorphic bisectional curvature [9]). For nearly Kähler manifolds we have the following result.

Theorem (6.2). *Let M be a compact non-Kähler nearly Kähler manifold such that the sectional curvature K is positive and the holomorphic bisectional curvature B is nonnegative. Then $H^{1,1}(M) = 0$.*

Proof. Let ξ be a form of bidegree (1, 1) on M; then $\xi(JX, JY) = \xi(X, Y)$ for all $X, Y \in \mathscr{X}(M)$. Without loss of generality we may assume that ξ is real. According to [7] there exists a local orthonormal frame field $\{E_1, \cdots, E_n, JE_1, \cdots, JE_n\}$ such that $\xi(E_i, JE_j) = 0$ for $i \neq j$.

In order to simplify the proof, we now introduce some classical tensor notation. We use the index convention that $1 \leq i, j, k, l \leq n$ and $1 \leq \alpha, \beta, \gamma, \delta \leq 2n$. Also we set $JE_i = E_{i^*}$ so that $n + 1 \leq i^*, j^*, k^*, l^* \leq 2n$. Define

$$\xi_{\alpha\beta} = \xi(E_\alpha, E_\beta) , \qquad \varphi_{\alpha\beta} = \nabla_{E_\alpha}(J)(E_\beta) ,$$
$$R_{\alpha\beta\gamma\delta} = \langle R_{E_\alpha E_\beta}E_\gamma, E_\delta\rangle , \qquad R_{\alpha\beta} = k(E_\alpha, E_\beta) ,$$
$$F(\xi) = \sum_{\alpha,\beta,\gamma} R_{\alpha\beta}\xi_{\alpha\gamma}\xi_{\beta\gamma} - \frac{1}{2}\sum_{\alpha,\beta,\gamma,\delta} R_{\alpha\beta\gamma\delta}\xi_{\alpha\beta}\xi_{\gamma\delta} .$$

It is well known that if ξ is harmonic and $F(\xi) \geq 0$, then $F(\xi) = 0$ and ξ is parallel. We have

$$\begin{aligned} F(\xi) &= 2\sum_{i,j} \{(R_{ijij} + R_{ij^*ij^*})\xi_{ii^*}{}^2 - R_{ii^*jj^*}\xi_{ii^*}\xi_{jj^*}\} \\ &= 2\sum_{i<j} \{R_{ii^*jj^*}(\xi_{ii^*} - \xi_{jj^*})^2 + 2\|\varphi_{ij}\|^2(\xi_{ii^*}{}^2 + \xi_{jj^*}{}^2)\} . \end{aligned}$$

Since $R_{ii^*jj^*} \geq 0$ for all i and j, it follows that $F(\xi) \geq 0$. Assume that ξ is harmonic; then ξ is parallel and $F(\xi) = 0$. Hence

$$R_{ii^*jj^*}(\xi_{ii^*} - \xi_{jj^*})^2 + 2\|\varphi_{ij}\|^2(\xi_{ii^*}{}^2 + \xi_{jj^*}{}^2) = 0 \tag{10}$$

for all i and j.

We wish to show that $\xi_{ii^*} = 0$ for all i. There are two cases.

Case 1. There exists j such that $\varphi_{ij} \neq 0$. Then (10) implies immediately that $\xi_{ii^*} = 0$.

Case 2. For all j we have $\varphi_{ij} = 0$. Then

$$R_{ii^*jj^*} = R_{ijij} + R_{ij^*ij^*} > 0 .$$

Hence from (10) it follows that $\xi_{ii^*} = \xi_{jj^*}$ for all j. By assumption M is not Kählerian, and so for some j and k, $\varphi_{jk} \neq 0$. By Case 1 we have $\xi_{jj^*} = 0$. Therefore $\xi_{ii^*} = 0$. This completes the proof of the theorem.

We remark that a slightly stronger result than Theorem (6.2) holds. Instead of assuming that M has positive sectional curvature, it is only necessary to suppose that $K_{xu} + K_{xJu} > 0$ for linearly independent x, u, Jx, Ju.

A modification of the proof of Theorem (6.2) also yields the following result.

Theorem (6.3). *Let M be a (compact) nearly Kähler manifold with nonnegative holomorphic bisectional curvature. Also, assume that $K_{xu} + K_{xJu} > 0$ for linearly independent x, u, Jx, Ju. Then $\dim H^{1,1}(M) = 1$ if M is Kählerian, and $H^{1,1}(M) = 0$ if M is not Kählerian.*

Furthermore we have the following theorem on holomorphic pinching.

Theorem (6.4). *Suppose M is a (compact) nearly Kähler manifold which satisfies condition $T(\rho, \sigma)$ with $\sigma > 0$, and assume M has holomorphic pinching $\delta \geq \frac{1}{2}(1 + \sigma)$. Then $\dim H^{1,1}(M) = 1$ if M is Kählerian, and $H^{1,1}(M) = 0$ if M is not Kählerian.*

Proof. This follows from Corollary (4.9) and Theorem (6.3).

7. Holomorphic forms on nearly Kähler manifolds

It is well known that a compact Kähler manifold with positive Ricci curvature has no holomorphic p-forms. In this section we give a generalization of this result to nearly Kähler manifolds. Where it is convenient, we use the notation of § 6.

Theorem (7.1). *Suppose M is a (compact) nearly Kähler manifold of pointwise constant type whose Ricci curvature k satisfies*

$$k(x, x) > \tfrac{1}{2}(p - 1)\|\nabla_x(J)(y)\|^2$$

for $x, y \in M_m$ with $\|x\| = \|y\| = 1$ and $\langle x, y\rangle = \langle Jx, y\rangle = 0$ for all $m \in M$. Then $H^{p,0}(M) = H^{0,p}(M) = 0$ for $p > 0$.

Proof. Let ξ be a real harmonic form which is the sum of complex forms of bidegrees $(p, 0)$ and $(0, p)$. It suffices to prove that $\xi = 0$. Let

$$\begin{aligned} F(\xi) &= \sum R_{\alpha\beta}\xi_{\alpha\alpha_2\cdots\alpha_p}\xi_{\beta\alpha_2\cdots\alpha_p} - \tfrac{1}{2}(p - 1) \sum R_{\alpha\beta\gamma\delta}\xi_{\alpha\beta\alpha_3\cdots\alpha_p}\xi_{\gamma\delta\alpha_3\cdots\alpha_p} \\ &= A - \tfrac{1}{2}(p - 1)B . \end{aligned}$$

We show that $F(\xi) \geq 0$. Assume the Ricci curvature is diagonalized with respect to the frame field $\{E_1, \cdots, E_n, JE_1, \cdots, JE_n\}$. Then, by Lemma (6.1),

$$\begin{aligned} A &= \sum (R_{ii} + R_{i^*i^*})(\xi_{ij\alpha_3\cdots\alpha_p}{}^2 + \xi_{ij^*\alpha_3\cdots\alpha_p}{}^2) \\ &= 2 \sum R_{ii}(\xi_{ij\alpha_3\cdots\alpha_p}{}^2 + \xi_{ij^*\alpha_3\cdots\alpha_p}{}^2) . \end{aligned}$$

Next we calculate B. We have by Lemmas (2.1) and (6.1) that

$$\begin{aligned} B &= 2 \sum \{(R_{ijkl} - R_{ijk^*l^*})\xi_{ij\alpha_3\cdots\alpha_p}\xi_{kl\alpha_3\cdots\alpha_p} \\ &\qquad + 2(R_{ijk^*l} + R_{ijkl^*})\xi_{ij\alpha_3\cdots\alpha_p}\xi_{k^*l\alpha_3\cdots\alpha_p} \\ &\qquad + (R_{i^*jk^*l} + R_{i^*jkl^*})\xi_{i^*j\alpha_3\cdots\alpha_p}\xi_{k^*l\alpha_3\cdots\alpha_p}\} \\ &= 2 \sum_{\alpha_3,\cdots,\alpha_p} \Big\| \sum_{i,j} (\xi_{ij\alpha_3\cdots\alpha_p}\varphi_{ij} + \xi_{i^*j\alpha_3\cdots\alpha_p}\varphi_{i^*j}) \Big\|^2 \\ &\leq 2 \sum_{\alpha_3,\cdots,\alpha_p} \sum_{i,j} (\xi_{ij\alpha_3\cdots\alpha_p}{}^2 + \xi_{ij^*\alpha_3\cdots\alpha_p}{}^2) \|\varphi_{ij}\|^2 . \end{aligned}$$

Hence

$$F(\xi) \geq \sum \{2R_{ii} - (p-1)\|\varphi_{ij}\|^2\}(\xi_{ij\alpha_3\cdots\alpha_p}{}^2 + \xi_{ij^*\alpha_3\cdots\alpha_p}{}^2) \geq 0 .$$

Therefore $F(\xi) = 0$ and we conclude that $\xi = 0$. Hence $H^{p,0}(M) = H^{0,p}(M) = 0$.

Theorem (7.2). *Suppose M is a (compact) nearly Kähler manifold of pointwise constant type α. Also, assume that M has holomorphic pinching δ such that*

$$\delta > \frac{(n-1)(2\alpha+1)}{6(n-1)\alpha + 3n + 1} \qquad \text{if } n \leq 5 ,$$

$$\delta > \frac{(n-1)\alpha + n - 3}{(n-1)(3\alpha+2)} \qquad \text{if } n > 5 ,$$

where $\dim M = 2n$. *Then $H^{p,0}(M) = H^{0,p}(M) = 0$ for $p > 0$.*

Proof. This follows from Proposition (4.12) and Theorem (7.1).

We now prove that under certain conditions the second cohomology group of M vanishes. We first note that the function F used in § 6 and § 7 is actually a quadratic form. The symmetric bilinear form associated with F is given on forms of degree 2 by the formula

$$F(\eta, \xi) = 2 \sum_{\alpha,\beta,\gamma} R_{\alpha\beta}\eta_{\alpha\gamma}\xi_{\beta\gamma} - \sum_{\alpha,\beta,\gamma,\delta} R_{\alpha\beta\gamma\delta}\eta_{\alpha\beta}\xi_{\gamma\delta} . \tag{11}$$

Proposition (7.3). *Suppose M is an almost Hermitian manifold with the property that $\langle R_{JwJx}Jy, Jz\rangle = \langle R_{wx}y, z\rangle$ for all tangent vectors w, x, y, z. Let η be a differential form of bidegree $(1,1)$ and ξ a differential form which is the sum of forms of bidegree $(2,0)$ and $(0,2)$. Then $F(\eta, \xi) = 0$.*

Proof. As usual we normalize η so that $\eta_{i\alpha} = 0$ for $\alpha \neq i^*$. It is then easy to verify that each of the sums in the right hand side of (11) vanishes.

The next two theorems are generalizations of results of [7].

Theorem (7.4). *Suppose that M is a compact nearly Kähler manifold of pointwise constant type α, and that the sectional curvature K of M satisfies*

$$K_{xu} \geq \alpha \,, \tag{12}$$

where $2n = \dim M$. Then $\dim H^2(M, \boldsymbol{R}) = 1$ if M is Kählerian, and $H^2(M, \boldsymbol{R}) = 0$ otherwise.

Proof. Assume M is not Kählerian, and let ω be a harmonic form of degree 2. We may write $\omega = \eta + \xi$, where η is of bidegree $(1, 1)$ and ξ is a sum of forms of bidegrees $(0, 2)$ and $(2, 0)$. Now (12) implies the hypotheses of Theorems (6.3) and (7.1) are satisfied. Even though η and ξ may not be harmonic, from the proofs of these two theorems we have $F(\eta) \geq 0$ and $F(\xi) \geq 0$. Hence by Proposition (7.3),

$$F(\omega) = F(\eta) + F(\xi) + 2F(\eta, \xi) = F(\eta) + F(\xi) \geq 0 \,.$$

We conclude that $F(\omega) = 0$ and so $F(\eta) = F(\xi) = 0$. Just as in the proofs of Theorems (6.3) and (7.1) we find that $\eta = \xi = 0$, and so $\omega = 0$.

A modification of the proof of Theorem (7.4) yields the following result.

Theorem (7.5). *Suppose M is a compact nearly Kähler manifold of pointwise constant type $\alpha > 0$. If M has holomorphicp inching $\delta \geq \frac{1}{2}(1 + \alpha)$, then $\dim H^2(M, \boldsymbol{R}) = 1$ for Kählerian M, and $H^2(M, \boldsymbol{R}) = 0$ otherwise.*

8. Einstein nearly Kähler manifolds of positive sectional curvature

Theorem (8.1). *Let M be a compact Einstein nearly Kähler manifold of pointwise constant type. If M has positive sectional curvature and nonnegative holomorphic bisectional curvature, then M is isometric either to complex projective space or to S^6.*

Since the proof is lengthy, we divide it into several lemmas. We shall frequently use the classical tensor notation of § 6; furthermore we continue to use the same index conventions. Our proof is patterned after the corresponding theorem for Kähler manifolds as given in [9].

Lemma (8.2). *Let M be an Einstein almost Hermitian manifold with $R_{ij} = \lambda g_{ij}$. Then*

$$\frac{1}{2} \sum_{\alpha} \nabla_\alpha \nabla_\alpha R_{11^*ii^*} = \sum_{\alpha, \beta} ({R_{1\alpha1^*\beta}}^2 - {R_{11^*\alpha\beta}}^2 - R_{1\alpha1\beta} R_{1^*\alpha1^*\beta}) + \lambda R_{11^*ii^*} \,. \tag{13}$$

This lemma is a special case of a formula of Berger in the Riemannian case [3, Lemma (6.2)]; the Riemann curvature tensors in Berger's paper differ from ours in sign.

Throughout the rest of this section, M will be a nearly Kähler manifold.

Furthermore we henceforth assume that a local orthonormal frame field $\{E_1, \cdots, E_n, JE_1, \cdots, JE_n\}$ has been chosen so that $R_{11^*i\alpha} = 0$ for $\alpha \neq i^*$. This choice is possible for nearly Kähler manifolds because the 2-form α_X defined by

$$\alpha_X(Y, Z) = \langle R_{XJX}Y, Z\rangle \qquad (X, Y, Z \in \mathscr{X}(M))$$

is of bidegree $(1, 1)$ by equation (2), and because of Corollary (4.5). Denote by Q_1 the right hand side of (13), and let $H_1 = R_{11^*11^*}$.

Lemma (8.3). *We have*

$$\begin{aligned}Q_1 \geq \sum_{i\geq 2} \{H_1R_{11^*ii^*} - 2R_{11^*ii^*}{}^2 + 2\|\varphi_{1i}\|^2H_1 - 2\|\varphi_{1i}\|^4\}\\ -4\sum_{2\leq i<j} \{\langle\varphi_{1i}, \varphi_{1j}\rangle^2 + \langle\varphi_{1i}, \varphi_{1j^*}\rangle^2\} .\end{aligned} \tag{14}$$

Proof. From Lemma (8.2) it follows that

$$\begin{aligned}
&Q_1 + H_1^2 - \lambda H_1 + 2\sum_{i\geq 2} R_{11^*ii^*}{}^2\\
&= \sum_{i,j\geq 2} \{R_{1i1^*j}{}^2 + R_{1i1^*j^*}{}^2 + R_{1i^*1^*j}{}^2 + R_{1i^*1^*j^*}{}^2 - R_{1i1j}R_{1^*i1^*j}\\
&\qquad - R_{1i1j^*}R_{1^*i1^*j^*} - R_{1i^*1j}R_{1^*i^*1^*j} - R_{1i^*1j^*}R_{1^*i^*1^*j^*}\}\\
&= \sum_{i,j\geq 2} \{(\langle\varphi_{1i}, \varphi_{1j^*}\rangle - R_{1i1j^*})^2 + 2R_{1i1j^*}R_{1i^*1j} + (\langle\varphi_{1i^*}, \varphi_{1j}\rangle - R_{1i^*1j})^2\\
&\qquad + (\langle\varphi_{1i}, \varphi_{1j}\rangle - R_{1i1j})^2 - 2R_{1i1j}R_{1i^*1j^*} + (\langle\varphi_{1i^*}, \varphi_{1j^*}\rangle - R_{1i^*1j^*})^2\}\\
&= \sum_{i,j\geq 2} \{(R_{1i1j^*} + R_{1i^*1j})^2 + (R_{1i1j} - R_{1i^*1j^*})^2 + 2\langle\varphi_{1i}, \varphi_{1j}\rangle^2 + 2\langle\varphi_{1i}, \varphi_{1j^*}\rangle^2\\
&\qquad - 2\langle\varphi_{1i}, \varphi_{1j}\rangle(R_{1i1j} + R_{1i^*1j^*}) - 2\langle\varphi_{1i}, \varphi_{1j^*}\rangle(R_{1i1j^*} - R_{1i^*1j})\}\\
&\geq -2\sum_{i,j\geq 2} \{\langle\varphi_{1i}, \varphi_{1j}\rangle^2 + \langle\varphi_{1i}, \varphi_{1j^*}\rangle^2 + \langle\varphi_{1i}, \varphi_{1j}\rangle R_{11^*ij} + \langle\varphi_{1i}, \varphi_{1j^*}\rangle R_{11^*ij^*}\}\\
&= -2\sum_{i,j\geq 2} \{\langle\varphi_{1i}, \varphi_{1j}\rangle^2 + \langle\varphi_{1i}, \varphi_{1j^*}\rangle^2\} .
\end{aligned}$$

Since $\lambda = H_1 + \sum_{i\geq 2} R_{11^*ii^*} + 2\sum_{i\geq 2} \|\varphi_{1i^*}\|^2$, the lemma follows.

Lemma (8.4). *Let M be a nearly Kähler Einstein manifold of (pointwise) constant type and nonnegative holomorphic bisectional curvature. Assume that the holomorphic sectional curvature H assumes its maximum on M at a unit vector $x \in M_m$, and that the local orthonormal frame field $\{E_1, \cdots, E_n, JE_1, \cdots, JE_n\}$ is chosen so that $H_1 = H(x)$. Then for $i = 2, \cdots, n$, we have*

$$Q_1 = \|\varphi_{1i}\|^2R_{11^*ii^*} = H_1 - 2R_{11^*ii^*} - \|\varphi_{1i}\|^2 = 0 . \tag{15}$$

Proof. Since H_1 is a maximum for H we have $Q_1 \leq 0$. Because M is of constant type the last sum in equation (14) vanishes. Therefore by Lemma

(8.3) and Corollary (3.2) we have

$$Q_1 \geq \sum_{i\geq 2} \{(R_{11^*ii^*} + 2\|\varphi_{1i}\|^2)(H_1 - 2R_{11^*ii^*} - \|\varphi_{1i}\|^2) + 5\|\varphi_{1i}\|^2 R_{11^*ii^*}\} \geq 0 . \tag{16}$$

Hence $Q_1 = 0$. The rest of (15) then follows from Corollary (3.2) and equation (16).

The following lemma generalizes a formula of [3].

Lemma (8.5). *Let M be a nearly Kähler manifold of real dimension $2n$. Then at each point $m \in M$ we have*

$$\frac{n(n+1)}{V(S^{2n-1})}\int_{S_m} H(x)dx = R(m) - 6\sum_{i<j} \|\nabla_{E_i}(J)(E_j)\|_m^2 , \tag{17}$$

where $V(S^{2n-1})$ is the volume of the unit sphere of dimension $2n-1$, dx is the canonical measure in the unit sphere S_m of the tangent space M_m, and $R(m)$ is the Ricci scalar curvature of M at m.

Proof. Let $\{e_1, \cdots, e_n, Je_1, \cdots, Je_n\}$ be a frame at m. Then for $x \in M_m$ write $x = \Sigma_{i=1}^n (a_i e_1 + b_i Je_i)$. A calculation shows that

$$\begin{aligned} H(x) = &\sum_i (a_i^2 + b_i^2)^2 R_{ii^*ii^*} + 2\sum_{i<j} (a_i^2 b_j^2 + a_j^2 b_i^2)(R_{ijij} + R_{iji^*j^*}) \\ &+ 2\sum_{i<j} (a_i^2 a_j^2 + b_i^2 b_j^2)(R_{ij^*ij^*} - R_{ij^*i^*j}) + 2\sum_{i<j} (a_i^2 + b_i^2)(a_j^2 + b_j^2) R_{ii^*jj^*} \\ &+ (\text{terms with at least one odd exponent}) . \end{aligned}$$

Now we have

$$\frac{1}{V(S^{2n-1})}\int_{S_m} a_i^4\, dx = \frac{3}{4n(n+1)} , \qquad \frac{1}{V(S^{2n-1})}\int_{S_m} a_i^2 a_j^2 = \frac{1}{4n(n+1)} (i \neq j),$$

and similarly for b_i^4, $b_i^2 b_j^2$ $(i \neq j)$, and $a_i^2 b_j^2$. Thus

$$\begin{aligned} \frac{1}{V(S^{2n-1})}\int_{S_m} H(x)\, dx &= \frac{2}{n(n+1)}\{\sum_i R_{ii^*ii^*} - \sum_{i<j} \|\varphi_{ij}\|^2 \\ &\qquad + \sum_{i<j} (R_{ijij} + R_{ij^*ij^*} + R_{ii^*jj^*})\} \\ &= \frac{1}{n(n+1)}\{R(m) - 6\sum_{i<j} \|\varphi_{ij}\|^2\} . \end{aligned}$$

Proof of Theorem (8.1). Since M is a nearly Kähler manifold of pointwise constant type, $\|\varphi_{ij}\|^2 = \|\varphi_{kl}\|^2$ for all i, j, k, l with $i \neq j$, $k \neq l$. By assumption M is compact and so the holomorphic sectional curvature H does, in fact, assume its maximum. Thus (15) holds, and so we have two cases.

Case 1. $\|\varphi_{ij}\|^2 = 0$ for all i and j. Then just as in [9] we find that M is isometric to complex projective space.

Case 2. $\|\varphi_{ij}\|^2 > 0$ for all $i \neq j$. From (15) it follows that $R_{11^*ii^*} = 0$ for $i \geq 2$ and that $H_1 = \|\varphi_{ij}\|^2$ for $i \neq j$. Furthermore, we have $\lambda = (2n-1)H_1$ and $R = 2n(2n-1)H_1$. On the other hand by Lemma (8.5) we have at any point $m \in M$ that

$$\frac{n(n+1)}{V(S^{2n-1})} \int_{S_m} H(x)\, dx = \{2n(2n-1) - 3n(n-1)\}H_1 = n(n+1)H_1 .$$

Therefore $H(x) = H_1$ for all $x \in M_m$, that is, M has constant holomorphic curvature H_1. Furthermore for unit vectors $x, u \in M_m$ with $\langle x, u\rangle = 0$, we have by Proposition (3.4) that

$$\begin{aligned} K_{xu} &= \frac{H_1}{4}(1 + 3\langle Jx, u\rangle^2) + \frac{3}{4}\|\nabla_x(J)(u)\|^2 \\ &= \frac{H_1}{4}(1 + 3\langle Jx, u\rangle^2) + \frac{3}{4}(1 - \langle Jx, u\rangle^2)H_1 = H_1 . \end{aligned}$$

Hence M has constant curvature. Since M is orientable, M is isometric to a sphere. In fact, M is isometric to S^6, because S^6 is the only sphere possessing a non-Kählerian almost complex structure.

An examination of the proof of Theorem (8.1) shows that actually a slightly stronger result holds. We state this as follows.

Theorem (8.6). *Let M be a compact Einstein nearly Kähler manifold of pointwise constant type with nonnegative holomorphic bisectional curvature. If M has the property that $K_{xy} + K_{xJy} > 0$ for linearly independent x, Jx, y, Jy, then M is isometric either to complex projective space or to S^6.*

We also have the following results.

Theorem (8.7). *Let M be a compact Einstein nearly Kähler manifold of global constant type α, and assume M has holomorphic pinching $\delta \geq \frac{1}{2}(1+\alpha)$. Then M is isometric either to complex projective space or to S^6.*

Proof. This follows from Corollary (4.9) and Theorem (8.6).

Theorem (8.8). *Let M be a compact nearly Kähler manifold with nonnegative holomorphic bisectional curvature, pointwise constant holomorphic curvature $\mu > 0$ and pointwise constant type α. Assume also that M has nonnegative holomorphic bisectional curvature. Then M is isometric either to complex projective space or to S^6.*

Proof. By Proposition (3.6) M is Einsteinian, and by Proposition (3.4) the sectional curvature of M is positive. Hence Theorem (8.8) follows from Theorem (8.1).

Theorems (8.6), (8.7), and (8.8) generalize results of [3] and [9].

9. The Chern classes of a nearly Kähler manifold

A well known theorem of Chern states that if Φ_{ij} is the matrix of (complex) curvature forms of a compact Kähler manifold M, then $\det(\delta_{ij}-\Phi_{ij}/(2\pi\sqrt{-1}))$ is the sum of differential forms which represent via de Rham's theorem the Chern classes of M. For example, see [16, p. 307]. In this section we find differential forms which represent the Chern classes of any compact almost Hermitian manifold M. In the case when M is nearly Kählerian, these formulas simplify slightly.

Theorem (9.1). *Let M be a compact almost Hermitian manifold with Riemannian connection ∇ and curvature operator $R_{XY}(X, Y \in \mathscr{X}(M))$. Define a tensor field S of type* (1, 3) *by*

$$\langle S_{WX}Y, Z\rangle = \tfrac{1}{2}\langle R_{WX}Y, Z\rangle + \tfrac{1}{2}\langle R_{WX}JY, JZ\rangle + \tfrac{1}{4}\langle \nabla_W(J)(Y), \nabla_X(J)(Z)\rangle - \tfrac{1}{4}\langle \nabla_W(J)(Z), \nabla_X(J)(Y)\rangle$$

for $W, X, Y, Z \in \mathscr{X}(M) \otimes C$. If $\{E_1, \cdots, E_n, JE_1, \cdots, JE_n\}$ is a local frame field on M, set

$$\Xi_{ij}(X, Y) = \langle S_{XY}E_i, E_j\rangle - \sqrt{-1}\,\langle S_{XY}E_i, JE_j\rangle$$

for $X, Y \in \mathscr{X}(M) \otimes C$ and $1 \le i, j \le n$. Then $\det(\delta_{ij} - \Xi_{ij}/(2\pi\sqrt{-1}))$ *is globally defined, and via de Rham's theorem it represents the total Chern class of M.*

Proof. We define a new connection D on M by $D_XY = \frac{1}{2}(\nabla_XY - J\nabla_XJY)$. Then $D_X(J)(Y) = 0$, and so D is a Hermitian connection in the sense of [16, p. 178] on the tangent bundle $\tau(M)$ of M, where $\tau(M)$ is viewed as a complex vector bundle on M. A calculation shows that S is the curvature operator determined by D, i.e., $S_{XY} = D_{[X,Y]} - [D_X, D_Y]$ for $X, Y \in \mathscr{X}(M)$. Then the matrix (Ξ_{ij}) is the curvature matrix defined by D on the complex vector bundle $\tau(M)$. Theorem (9.1) now follows from [16, Theorem 3.1, p. 307].

Corollary (9.2). *If M is a compact nearly Kähler manifold, then the total Chern class is* $\det(\delta_{ij} - \Xi_{ij}/(2\pi\sqrt{-1}))$ *where*

$$\Xi_{ij}(X, Y) = \langle S_{XY}E_i, E_j\rangle - \sqrt{-1}\langle S_{XY}E_i, JE_j\rangle\ ,$$
$$\langle S_{WX}Y, Z\rangle = \langle R_{WX}Y, Z\rangle - \tfrac{1}{2}\langle \nabla_W(J)(X), \nabla_Y(J)(Z)\rangle + \tfrac{1}{4}\langle \nabla_W(J)(Y), \nabla_X(J)(Z)\rangle - \tfrac{1}{4}\langle \nabla_W(J)(Z), \nabla_X(J)(Y)\rangle\ ,$$

for $W, X, Y, Z \in \mathscr{X}(M) \otimes C$.

Corollary (9.3). *If M is a compact nearly Kähler manifold, then the first Chern class γ_1 of M is given by*

$$\gamma_1(X, Y) = \frac{1}{2\pi}\sum_{i=1}^{n}\left\{\langle R_{XY}E_i, JE_i\rangle - \frac{1}{2}\langle \nabla_X(J)(E_i), J\nabla_Y(J)(E_i)\rangle\right\}$$

for $X, Y \in \mathscr{X}(M)$, *where* $\{E_1, \cdots, E_n, JE_1, \cdots, JE_n\}$ *is a local frame field on* M. *Hence, for* $X \in \mathscr{X}(M)$,

$$2\pi\gamma_1(X, JX) = k(X, X) + \frac{3}{2}\sum_{i=1}^{n} \|\nabla_X(J)(E_i)\|^2 .$$

10. Immersions of nearly Kähler manifolds

Recall [10] that an almost Hermitian manifold is said to be *quasi-Kählerian* provided that $\nabla_X(J)(Y) + \nabla_{JX}(J)(JY) = 0$ for all $X, Y \in \mathscr{X}(M)$. A nearly Kähler manifold is quasi-Kählerian [10]. Furthermore in [10] it is shown that an almost Hermitian submanifold M of a quasi-Kählerian (nearly Kählerian) manifold $\overline{M}$ is itself quasi-Kählerian (nearly Kählerian) and is a minimal variety. Moreover, the following is true.

Proposition (10.1). *Let* M *be an almost Hermitian submanifold of* $\overline{M}$, *and denote by* B *and* $\overline{B}$ *the respective holomorphic bisectional curvatures. If* $\overline{M}$ *is quasi-Kählerian, then* $B_{XY} \leq \overline{B}_{XY}$ *for all* $X, Y \in \mathscr{X}(M)$.

Proof. Let T denote the configuration tensor of M in $\overline{M}$ (see [10]). Then [10] we have $T_X Y + T_{JX} JY = 0$ for all $X, Y \in \mathscr{X}(M)$. The Gauss equation [10] asserts that, for $W, X, Y, Z \in \mathscr{X}(M)$,

$$\langle R_{WX}Y, Z\rangle = \langle T_W Y, T_X Z\rangle - \langle T_W Z, T_X Y\rangle + \langle \overline{R}_{WX}Y, Z\rangle ,$$

where R_{XY} and $\overline{R}_{XY}$ are the curvature operators of M and $\overline{M}$ respectively. From the Gauss equation it follows that

$$\begin{aligned}\langle R_{XJX}Y, JY\rangle - \langle \overline{R}_{XJX}Y, JY\rangle &= \langle T_X Y, T_{JX}JY\rangle - \langle T_X JY, T_{JX}Y\rangle \\ &= -\|T_X Y\|^2 - \|T_X JY\|^2 .\end{aligned}$$

Hence the proposition follows.

This generalizes a result of [9]. Next we generalize two theorems of F. Frankel [8] to nearly Kähler manifolds (see also [9]).

Theorem (10.2). *Let* M *be a compact connected nearly Kähler manifold whose sectional curvature satisfies*

$$K_{xy} + K_{Jxy} > \|\nabla_x(J)(y)\|^2 \tag{18}$$

for all $x, y \in M_m$ *with* $\|x\| = \|y\| = 1$ *and* $\langle x, y\rangle = \langle Jx, y\rangle = 0$ *for all* $m \in M$. *If* V *and* W *are compact almost Hermitian submanifolds of* M *such that* $\dim V + \dim W \geq \dim M$, *then* V *and* W *have a nonempty intersection.*

Proof. Assume that $V \cap W$ is empty. Let σ be a unit speed shortest geodesic from V to W. Assume that σ is defined on $[0, b]$, and let $\sigma(0) = p \in V$ and $\sigma(b) = q \in W$. Since the first variation of arc length vanishes at σ, it follows that $\sigma'(0)$ is normal to V at p and $\sigma'(b)$ is normal to W at q.

Since $\dim V + \dim W \geq \dim M$, there exists a vector field $X \in \mathscr{X}(M)$ which is parallel along σ and tangent to both V and W at p and q, respectively. Then JX, although it may not be parallel along σ, is tangent to both V and W at p and q, respectively. Furthermore $\langle X, \sigma' \rangle(t) = \langle JX, \sigma' \rangle(t) = 0$ for $0 \leq t \leq b$.

Let S and T denote the configuration tensors of V and W. The second variation of arc length with respect to the infinitesimal variations X and JX is given as follows.

$$L_X''(0) = \langle T_X X, \sigma' \rangle(b) - \langle S_X X, \sigma' \rangle(0) - \int_0^b \langle R_{X\sigma'} X, \sigma' \rangle(t) dt ,$$

$$L_{JX}''(0) = \langle T_{JX} JX, \sigma' \rangle(b) - \langle S_{JX} JX, \sigma' \rangle(0) + \int_0^b \{\| \nabla_{\sigma'}(JX) \|^2 - \langle R_{JX\sigma'} JX, \sigma' \rangle\}(t)\, dt .$$

We have

$$L_X''(0) + L_{JX}''(0) = \int_0^b \{\| \nabla_{\sigma'}(J)(X) \|^2 - (K_{X\sigma'} + K_{JX\sigma'}) \| X \|^2\}(t)\, dt < 0.$$

Hence at least one of $L_X''(0)$ and $L_{JX}''(0)$ is negative. This contradicts the assumption that σ is a shortest geodesic from V to W. Hence the theorem follows.

The above proof is patterned after the corresponding result for Kähler manifolds as proved in [9].

Theorem (10.3). *Let N be a compact nearly Kähler manifold whose sectional curvature satisfies* (18). *Then every holomorphic correspondence of N has a fixed point.*

Proof. We set $M = N \times N$, $V =$ diagonal $(N \times N)$, and let W be the holomorphic correspondence (which is just an almost Hermitian submanifold of $N \times N$). We must modify the proof of Theorem (10.2) in order to show that V and W intersect.

Suppose $V \cap W$ is empty. Clearly the proof of Theorem (10.2) will carry over provided we can show the sectional curvature of $N \times N$ satisfies (18) at some point of σ. Consider the vector fields σ', X and JX defined along σ. Write

$$\sigma' = \sigma_1' \oplus \sigma_2' , \quad X = X_1 \oplus X_2 , \quad JX = JX_1 \oplus JX_2 ,$$

where σ_1', X_1, JX_1 are tangent to the first factor of $N \times N$ and σ_2', X_2, JX_2 are tangent to the second factor. Now σ is normal to V, and X and JX are tangent to V for $t = 0$. Hence we have

$$\sigma_1'(0) = -\sigma_2'(0) , \qquad X_1(0) = X_2(0) , \qquad JX_1(0) = JX_2(0) .$$

It follows that $\sigma_1'(0)$, $\sigma_2'(0)$, $X_1(0)$, $X_2(0)$, $JX_1(0)$, $JX_2(0)$ are all nonzero. Therefore for $t = 0$ we have

$$K_{X_\sigma'} + K_{JX_\sigma'} \geq K_{X_1\sigma_1'} + K_{JX_1\sigma_1'} > 0 .$$

Thus the proof is complete.

In [11] we proved that S^6 with the usual almost complex structure has no 4-dimensional almost complex submanifolds. A generalization of this is proved in [13]. We now give another generalization, this time for nearly Kähler manifolds of constant type. First we prove a lemma.

Lemma (10.4). *Let $\overline{M}$ be a nearly Kähler manifold, and M an almost Hermitian submanifold with $\dim \overline{M} - \dim M = 2$. Denote by T the configuration tensor of M in $\overline{M}$. Then for each $m \in M$ there exists $x \in M_m$ with $\|x\| = 1$ such that $T_x y = 0$ for all $y \in M_m$ with $\langle x, y\rangle = \langle Jx, y\rangle = 0$.*

Proof. If $T_x x = 0$ for all $x \in M_m$, the lemma is clear. Otherwise, let x be the point on the unit sphere of M_m at which the function $y \to \|T_y y\|^2$ assumes its maximum. Since the first derivative of $y \to \|T_y y\|^2$ vanishes at x, we have $\langle T_x x, T_x y\rangle = \langle T_x x, T_x Jy\rangle = 0$ whenever $\langle x, y\rangle = \langle Jx, y\rangle = 0$. Furthermore, let $u = \frac{1}{\sqrt{2}}(x + Jx)$. Then $T_u u = JT_x x$, and so $y \to \|T_y y\|^2$ also achieves its maximum on the unit sphere at u. Hence

$$0 = \langle T_u u, T_u y\rangle = \frac{1}{\sqrt{2}} \langle JT_x x, T_x y + T_x Jy\rangle .$$

Replacing y by Jy in this equation and subtracting the result we obtain $\langle JT_x x, T_x y\rangle = \langle JT_x x, T_x Jy\rangle = 0$. Thus $T_x y$ and $T_x Jy$ are perpendicular to both $T_x x$ and $JT_x x$. Since $T_x x$ and $JT_x x$ span $M_m^\perp$, we have $T_x y = T_x Jy = 0$.

Theorem (10.5). *Let $\overline{M}$ be a non-Kähler nearly Kähler manifold of pointwise constant type, and M an almost Hermitian submanifold of $\overline{M}$ which is Kählerian. Then $\dim M \leq \dim \overline{M} - 4$.*

Proof. Assume $\dim M = \dim \overline{M} - 2$. By Lemma (10.4) for each $m \in M$ there exists $x \in M_m$ such that $T_x y = 0$ for all $y \in M_m$ with $\langle x, y\rangle = \langle Jx, y\rangle = 0$. This implies that $\|\overline{\nabla}_x(J)(y)\|^2 = \|\nabla_x(J)(y)\|^2 = 0$, which is impossible. Hence the theorem follows.

In [14] we proved that any 6-dimensional nearly Kähler manifold which is a submanifold of R^8, and whose almost complex structure is derived from a 3-fold vector cross product on R^8 has pointwise constant type. Thus Theorem (10.5) applies to these manifolds.

The next theorem, in contrast to Theorem (10.5) and the results of [11] and [13], shows that in different circumstances Kähler manifolds arise quite frequently as almost Hermitian submanifolds of nearly Kähler manifolds.

Theorem (10.6). *Let M be a nearly Kähler manifold. For each $m \in M$ set*

$$\mathscr{K}(m) = \{x \in M_m \mid \nabla_x(J)(y) = 0 \text{ for all } y \in M_m\} .$$

Then on any open subset of M on which dim $\mathscr{K}(m)$ *is constant, the distribution* $m \to \mathscr{K}(m)$ *is integrable. Furthermore the integral submanifolds are Kähler submanifolds of M.*

Proof. Let W and X be vector fields which at each point lie in the distribution $m \to \mathscr{K}(m)$. We have, for $Y, Z \in \mathfrak{X}(M)$,

$$\begin{aligned}\langle \nabla_{[W,X]}(J)(Y), Z\rangle &= \langle R_{WX}(J)(Y), Z\rangle + \langle [[\nabla_W, \nabla_X], J]Y, Z\rangle \\ &= \langle R_{WX}JY, Z\rangle + \langle R_{WX}Y, JZ\rangle \\ &= \langle \nabla_W(J)(X), \nabla_{JY}(J)(Z)\rangle = 0 .\end{aligned}$$

Hence $[W, X]$ lies in the distribution $m \to \mathscr{K}(m)$. Therefore, by the Frobenius theorem, it follows that $m \to \mathscr{K}(m)$ is integrable on open sets of M on which dim $\mathscr{K}(m)$ is constant.

Let M' be an integral submanifold of $m \to \mathscr{K}(m)$. Then M' is a Riemannian submanifold of M, and it also is easy to verify that M' is an almost complex submanifold of M. Denote by δ and t the connection and configuration tensor of M'. For $m \in M'$ and $x, y \in M'_m(=\mathscr{K}(m))$ we have

$$0 = \nabla_x(J)(y) = \delta_x(J)(y) + t_xJy - Jt_xy .$$

Hence $\delta_x(J)(y) = 0$, and so M' is Kählerian.

Bibliography

[1] M. Berger, *Sur quelques variétés riemanniens suffisament pincees,* Bull. Soc. Math. France **88** (1960) 55–71.

[2] ——, *Pincement riemannien et pincement holomorphe,* Ann. Scuola Sup. Pisa **14** (1960) 151–159.

[3] ——, *Sur les variétés d'Einstein compactes,* C. R. III^e Reunion Math. Expression Latine, Namur (1965) 35–55.

[4] R. L. Bishop & S. I. Goldberg, *On the topology of positively curved Kähler manifolds,* Tôhoku Math. J. **15** (1963) 359–364.

[5] ——, *On the topology of positively curved Kähler manifolds.* II, Tôhoku Math. J. **17** (1965) 310–318.

[6] ——, *Some implications of the generalized Gauss-Bonnet theorem,* Trans. Amer. Math. Soc. **112** (1964) 508–535.

[7] ——, *On the second cohomology group of a Kähler manifold of positive curvature,* Proc. Amer. Math. Soc. **16** (1965) 119–122.

[8] T. Frankel, *Manifolds with positive curvature,* Pacific J. Math. **11** (1961) 165–174.

[9] S. I. Goldberg & S. Kobayashi, *Holomorphic bisectional curvature,* J. Differential Geometry **1** (1967) 225–233.

[10] A. Gray, *Minimal varieties and almost Hermitian submanifolds,* Michigan Math. J. **12** (1965) 273–287.

[11] ——, *Almost complex submanifolds of the six sphere,* Proc. Amer. Math. Soc. **20** (1969) 277–279.

[12] ——, *Vector cross products on manifolds,* Trans. Amer. Math. Soc. **141** (1969) 465–504.

[13] ——, *Kähler submanifolds of homogeneous almost Hermitian manifolds,* Tôhoku Math. J. **21** (1969) 190–194.

[14] ——, *Six-dimensional almost complex manifolds defined by means of three-fold vector cross products,* Tôhoku Math. J., **21** (1969) 614–620.

[15] S. Kobayashi, *On compact Kähler manifolds with positive definite Ricci tensor,* Ann. of Math. **74** (1961) 570–574.
[16] S. Kobayashi & K. Nomizu, *Foundations of differential geometry,* Vol. **II**, John Wiley, New York, 1969.
[17] J. L. Kozul, *Variétés Kählerienes,* Notes, São Paulo, 1957.
[18] Y. Tsukamoto, *On Kählerian manifolds with positive holomorphic curvature,* Proc. Japan Acad. **33** (1957) 333–335.
[19] J. Wolf & A. Gray, *Homogeneous spaces defined by Lie group automorphisms.* I, J. Differential Geometry **2** (1968) 77–114.
[20] ——, *Homogeneous spaces defined by Lie group automorphisms.* II, J. Differential Geometry **2** (1968) 115–159.

UNIVERSITY OF MARYLAND